

Discrete Mathematics

Lecture 11: Graph Theory

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Scribes: Definition and ...

11.1 Definition of Graph and its terminology

Definition : Graph

A **graph** G consists of two parts:

- A set $V = V(G)$ whose elements are called *vertices, points, or nodes*.
- A collection $E = E(G)$ of unordered pairs of distinct vertices called *edges*.

We write $G(V, E)$ or $G = (V, E)$ when we want to emphasize the two parts of G .

• multigraph

A **multigraph** $G = G(V, E)$ also consists of a set V of vertices and a set E of edges except that E may contain *multiple* edges, i.e., edges connecting the same endpoints, and E may contain one or more *loops*, i.e., an edge whose endpoints are the same vertex.

Example. Describe formally the following graph.

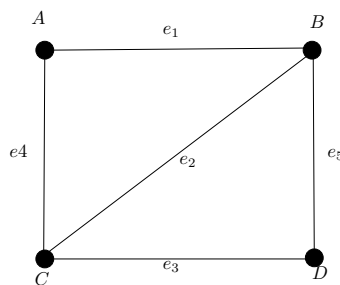


Figure 11.1:

• adjacency and incidence in a graph G :

Suppose $e = \{u, v\}$ is an edge in G , i.e., u and v are *endpoints* of e . Then the vertex u is said to be *adjacent* to the vertex v , and the edge e is said to be *incident* on u and on v .

- **Path** and its **length** in a graph(multigraph) G .

A *path* α in G with *origin* v_0 and *end* v_n is an alternating sequence of vertices and edges of the form

$$v_0, e_1, v_1, e_2, \dots, e_{n-1}, v_{n-1}, e_n, v_n$$

where each edge e_i is incident on vertices v_{i-1} and v_i . The number n of edges is called the *length* of α . When there is no ambiguity, we denote α by its sequence of edges, $\alpha = (e_1, e_2, \dots, e_n)$, or by its sequence of vertices, $\alpha = (v_0, v_1, \dots, v_n)$.

- **walk, length, closed walk, open walk.**

- If no edge in the $x - y$ walk is repeated, then the walk is called an *$x-y$ trail*. A closed $x - x$ trail is called *circuit*.

- **Simple path** and a **trail** in a graph(multigraph) G .

A path $\alpha = (v_0, v_1, \dots, v_n)$ is *simple* if all the vertices are distinct. The path is a *trail* if all the edges are distinct.

- **Closed** and **cycle** in G :

a path $\alpha = (v_0, v_1, \dots, v_n)$ is *closed* if $v_0 = v_n$, that is, if origin $(\alpha) = \text{end}(\alpha)$. The path α is a *cycle* if it is closed and if all vertices are distinct except $v_0 = v_n$. A cycle of length k is called a *k -cycle*. A cycle in a graph must therefore have length three or more.

Repeated Vertex(-ices)	Repeated Edge(s)	Open	Closed	Name
Yes	Yes	Yes		Walk(open)
Yes	Yes		Yes	Walk(closed)
Yes	No	Yes		Trail
Yes	No		Yes	Circuit
No	No	Yes		Path
No	No		Yes	Cycle

- **distance :**

Let u and v be vertices in a graph G . The distance $d(u, v)$ between u and v is equal to the length of a shortest path between u and v if $u \neq v$. The distance is 0 if $u = v$, i.e., $d(u, v) = 0$.

Example : Find (i) all simple paths from A to C and (ii) $d(A, C)$ in Fig 11.2.

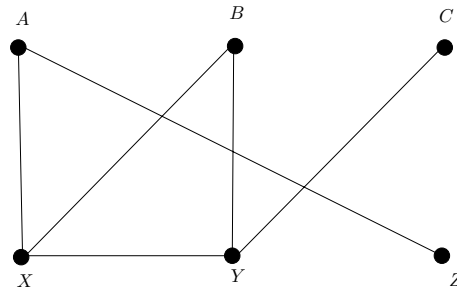


Figure 11.2:

iii) Find all cycles.

- **Connected graph(multigraph)** : a graph(multigraph) G is *connected* if there is a path between any two of its vertices.
- The *diameter* of G , written $\text{diam}(G)$, is the maximum distance between any two of its vertices.

Example : Find diameter of the connected graph G in Fig 11.2.

• Connected Components

Let G be a graph(multigraph). A *connected component* of G is a subgraph of G which is not contained in any larger connected subgraph of G , and denoted by $\kappa(G)$.

Example : Find the connected components of the graph G in Fig 11.3.

• **Definition** : Let V be a finite nonempty set. We say that the pair (V, E) determines a *multigraph* G with vertex set V and edges set E if, for some $x, y \in V$, there are two or more edges in E of the form (a) (x, y) (for a directed multigraph), or (b) $\{x, y\}$ (for an undirected multigraph). In either case, we write $G = (V, E)$ to designate the multigraph, just as we did for graphs.

If there are three edges from a to b , we say that the edge (a, b) has *multiplicity* 3.

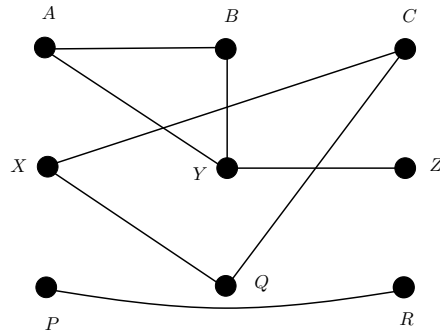


Figure 11.3:

11.2 Representation of Graph

- Adjacency matrix. Let $A = [a_{ij}]$ be the $m \times m$ matrix defined by

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \text{ is an edge, i.e., if } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

Then A is called the *adjacency matrix* of G .

- Incidence matrix. Let $M = [m_{ij}]$ be the $m \times m$ matrix defined by

$$m_{ij} = \begin{cases} 1, & \text{if vertex } v_i \text{ is incident on the edge } e_j; \\ 0, & \text{otherwise.} \end{cases}$$

Then M is called the *incidence matrix* of G .

- The *linked (adjacent) list representation* of G shows each vertex u of G followed by its set of adjacent vertices.

Example

11.3 Subgraphs, Complements, and Graph Isomorphism

Definition

- Let G be a graph. Then H is a *subgraph* of G if $V(H) \subseteq V(G)$, i.e., the vertices of H are also vertices of G , and $E(H) \subseteq E(G)$, i.e., the edges of H are also edges of G . In other words, $H(V', E')$ is a subgraph of $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.
- Suppose $H = H(V', E')$ is a subgraph of $G = G(V, E)$. Then H is called a *full subgraph* of G if E' contains all the edges of E whose endpoints lie in V' . In this case H is called the subgraph of G *generated* by V' .
- Given a (directed or undirected) graph $G = (V, E)$, let $G_1 = (V_1, E_1)$ be a subgraph of G . If $V_1 = V$, then G_1 is called a *spanning subgraph* of G .

Example

Which one is subgraph and spanning subgraph in Fig 11.4? How many spanning subgraphs are there ?

Definition : Let V be a set of n vertices. The *complete graph* on V , denoted K_n , is a loop-free undirected graph, where for all $a, b \in V$, $a \neq b$, there is an edge $\{a, b\}$.

Definition : Let G be a loop-free undirected graph on n vertices. The *complement of G* , denoted \bar{G} , is the subgraph of K_n consisting of the n vertices in G and all edges that are not in G . (If $G = K_n$, \bar{G} is a graph consisting of n vertices and no edges. Such a graph is called a *null* graph.)

Definition : Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. A function $f : V_1 \rightarrow V_2$ is called a *graph isomorphism* if (a) f is one-to-one and onto, and (b) for all $a, b \in V_1$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$. When such a function exists, G_1 and G_2 are called *isomorphic graphs*.

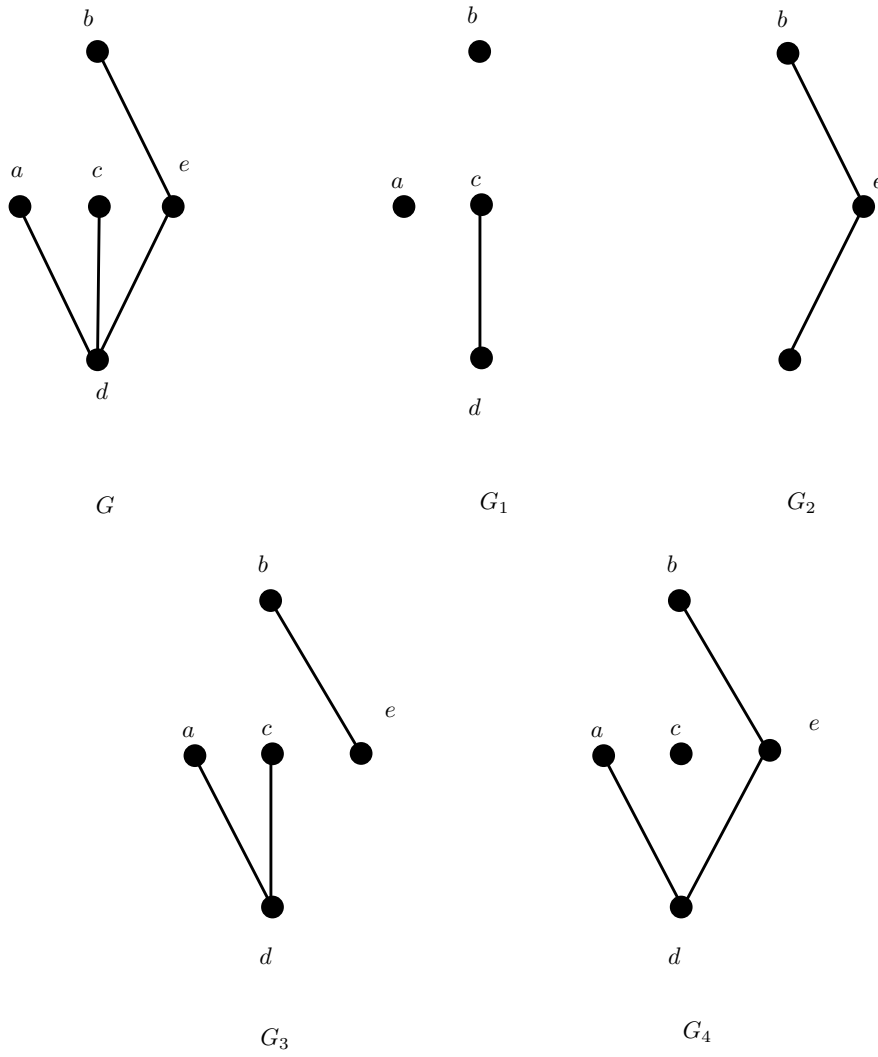


Figure 11.4:

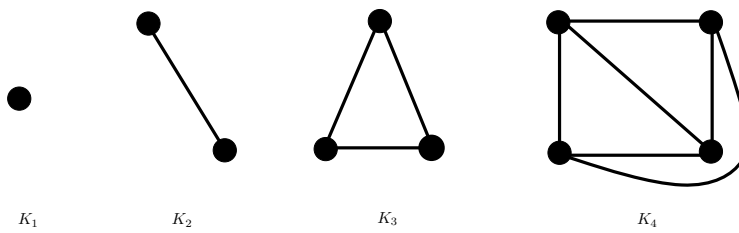


Figure 11.5:

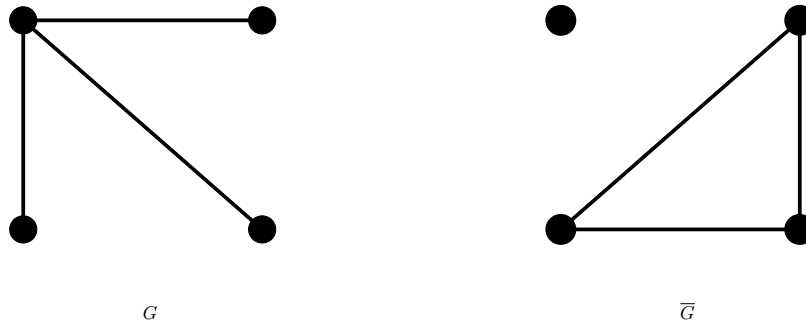


Figure 11.6:

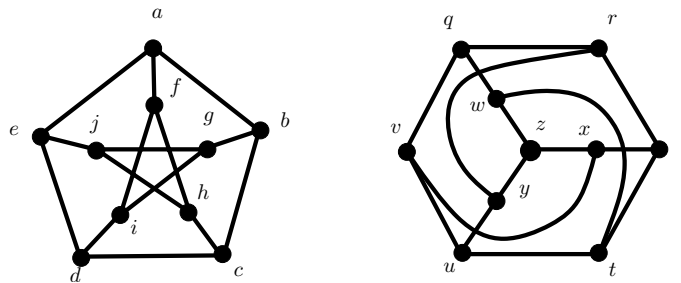


Figure 11.7:

One finds that the correspondence given by

$$\begin{aligned}
 a \rightarrow q & \quad c \rightarrow u & e \rightarrow r & \quad g \rightarrow x & i \rightarrow z \\
 b \rightarrow v & \quad d \rightarrow y & f \rightarrow w & \quad h \rightarrow t & j \rightarrow s
 \end{aligned}$$

Definition : Given any graph G , we can obtain a new graph by dividing an edge of G with additional vertices. Two graphs G and G' are said to be *homeomorphic* if they can be obtained from isomorphic graphs by this method.

Example The graph (a) and (b) are not isomorphic; but they are homeomorphic since

each can be obtained from (c) by adding appropriate vertices.

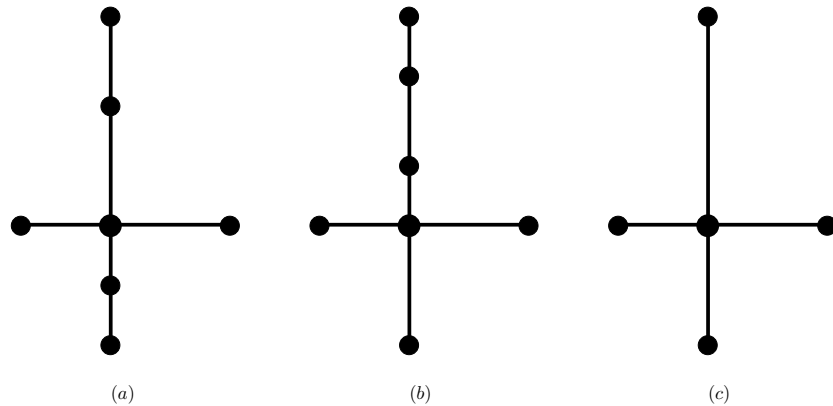


Figure 11.8:

11.4 Vertex Degree: Euler Trails and Circuits

Definition : The *degree* of a vertex v in a graph G , written $\deg(v)$, is equal to the number of edges which are incident on v or, in other words, the number of edges which contain v as an endpoint.

If a $\deg(v)=1$, it is called a *pendant* vertex.

Theorem : The sum of the degrees of the vertices of a graph is equal to twice the number of edges. ($\sum_{v \in V} \deg(v) = 2|E|$)

- An undirected graph (or multigraph) where each vertex has the same degree is called a *regular graph*. If $\deg(v) = k$ for all vertices v , then the graph is called *k-regular*.

Definition : Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G is said to have an *Euler circuit* if there is a circuit in G that traverses every edge of the graph exactly once. If there is an open trail from a to b in G and this trail traverses each edge in G exactly once, the trail is called an *Euler trail*.

- can be drawn without any breaks in the curve and without repeating any edge.

Theorem : Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G has an Euler circuit if and only if G is connected and every vertex in G has even degree.

Corollary : If G is an undirected graph or multigraph with no isolated vertices, then we can construct an Euler trail in G if and only if G is connected and has exactly two vertices of odd degree.

Definition : Let $G = (V, E)$ be a directed graph or multigraph. For each $v \in V$,

- The *incoming*, or *in*, *degree* of v is the number of edges in G that are incident into v , and this is denoted by $id(v)$.
- The *outgoing*, or *out*, *degree* of v is the number of edges in G that are incident from v , and this is denoted by $od(v)$.

Theorem : Let $G = (V, E)$ be a directed graph or multigraph with no isolated vertices. The graph G has a directed Euler circuit if and only if G is connected and $id(v) = od(v)$ for all $v \in V$.

Theorem : Euler formula Let $G = (V, E)$ (connected map) be a undirected graph with V vertices, E edges, and F faces. Then

$$V - E + F = 2$$

Proof : Suppose the connected graph G consist of a single vertex P as Fig 11.9(a). Then $V = 1$ and $E = 0$ and there is one face, i.e., $F = 1$. Thus in this case $V - E + F = 2$. Otherwise G can be built up from a single vertex by the following two constructions:

1. Add a new vertex q_2 and connect it to an existing vertex q_1 by an edge which does not cross any existing edge, as in Fig 11.9(b).
2. Connect two existing vertices q_1 and q_2 by an edge e which does not cross any existing edge, as in Fig 11.9(c).

The first operation does not change the value of $V - E + F$ since both V and E increased by 1, but the number R of regions is not changed. The second operation also does not change the value of $V - E + F$ since V does not change, E is increased by 1, and it can be shown

that the number F of regions(faces) is also increased by 1. Accordingly, G must have the same value of $V - E + F$ as the face consisting of a single vertex; that is, $V - E + F = 2$, and the theorem is proved.

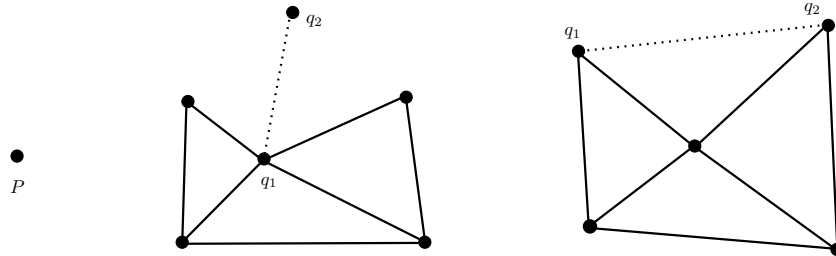


Figure 11.9:

Examples :

- The number vertices and the number of edges for four connected planar multigraphs as given below. Determine the number R of regions each graph must have.
 (a) $V = 10, E = 14$ (b) $V = 6, E = 7$ (c) $V = 25, E = 60$, (d) $V = 14, E = 12$
- Draw three connected maps of a graph with four vertices and five regions(faces)
 $E = V + F - 2 = 4 + 5 - 2 = 7$
- Draw three connected map of a graph with six vertices and seven edges.
 $F = E - V + 2 = 7 - 6 + 2 = 3$

11.5 Special Graphs

- **Complete Graph K_n .**
- **Regular graph, k - regular**
- **Eulerian graph** : if there exists a closed traversable trail, called an eulerian trail.

Example : The number of m of edges in the complete graph K_n . Each pair of vertices determines an edge. Thus $m = C(n, 2) = n(n - 2)/2$ since there are $C(n, 2)$ ways of selecting two vertices out of n vertices.

$$|E(K_8)| = 28, |E(K_{12})| = 66, |E(K_{15})| = 105$$

- hypercube graph

Definition : A graph (or multigraph) G is called *planar* if G can be drawn in the plane with its edges intersecting only at vertices of G . Such a drawing of G is called *embedding* of G in the plane.

Example

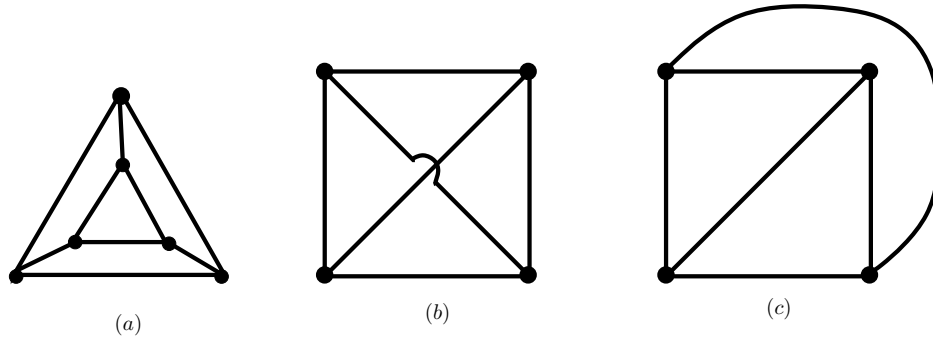


Figure 11.10:

Example K_1, K_2, K_3 , and K_4 are planar. If K_5 is planar, then any embedding would have to contain the pentagon.

Definition : A graph $G = (V, E)$ is called *bipartite* if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$, and every edge of G is of the form $\{a, b\}$ with $a \in V_1$ and $b \in V_2$. If each vertex in V_1 is joined with every vertex in V_2 , we have a *complete bipartite graph*. In this case, if $|V_1| = m, |V_2| = n$, the graph is denoted by $K_{m,n}$.

Example : Q_1, Q_2 , and Q_3 are bipartite. And Q_n is also bipartite.

Theorem : Let $G = (V, E)$ be a connected planar graph or multigraph with $|V| = v$ and $|E| = e$. Let r be the number of regions in the plane determined by a planar embedding (or, depiction) of G ; one of these regions has infinite area and is called *the infinite region*. Then $v - e + r = 2$.

Proof

Corollary : Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v, |E| = e >$

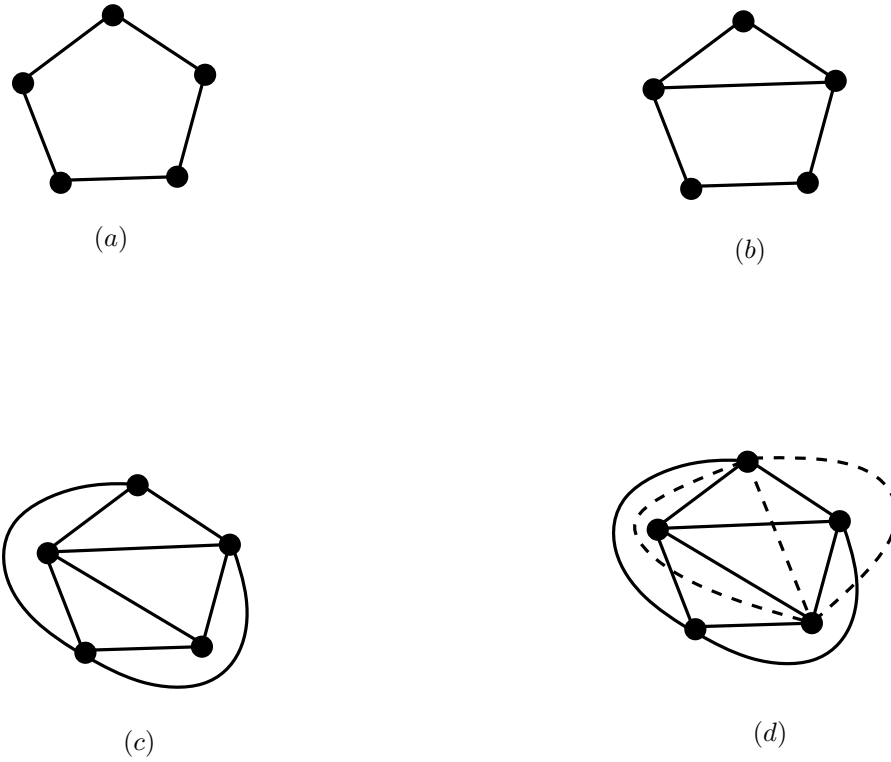


Figure 11.11:

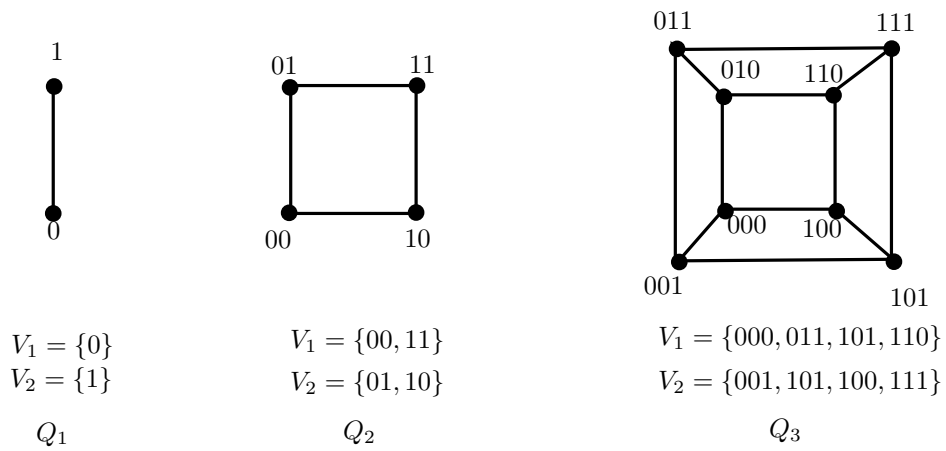


Figure 11.12:

2, and r regions. Then $3r \leq 2e$ and $e \leq 3v - 6$.

Proof: Since G is loop-free and is not a multigraph, the boundary of each region (including the infinite region) contains at least three edges – hence, each region has degree ≥ 3 . Consequently, $2e = 2|E| =$ the sum of the degrees of the r regions determined by G and $2e \geq 3t$. From Euler's Theorem, $2 = v - e + r \leq v - e + (2/3)e = v - (1/3)e$, so $6 \leq 3v - e$, or $e \leq 3v - 6$.

- If $G = (V, E)$ is a loop-free connected graph with $|E| > 2$, then if $e > 3v - 6$, it follows that G is not planar. However, if $e \leq 3v - 6$, we cannot conclude that G is planar.

Example: The graph K_5 is loop-free and connected with ten edges and five vertices. Consequently, $3v - 6 = 15 - 6 = 9 < 10 = e$. Therefore, by Corollary, we find that K_5 is nonplanar.

Example: The graph $K_{3,3}$ is loop-free and connected with nine edges and six vertices. Here $3v - 6 = 18 - 6 = 12 \geq 9 = e$. It would be a mistake to conclude from this that $K_{3,3}$ is planar. It would be the mistake of arguing by the converse.

However, $K_{3,3}$ is nonplanar. If $K_{3,3}$ were planar, then since each region in the graph is bounded by at least four edges, we have $4r \leq 2e$. (We found a similar situation in the proof of Corollary.) From Euler's Theorem, $v - e + r = 2$, or $r = e - v + 2 = 9 - 6 + 2 = 5$, so $20 = 4r \leq 2e = 18$. From this contradiction we have $K_{3,3}$ being nonplanar.

11.6 Colored Graphs

- *Vertex coloring* or *coloring* of a graph G , and the **chromatic number** of G , denoted by $\chi(G)$.

A *coloring* of a graph G is an assignment of colors to the vertices of G such that adjacent vertices have different colors. We say that G is *n-colorable* if there exists a coloring of G which uses n colors. Then minimum number of colors needed to paint G is called the *chromatic number* of G .

Example : Determine the chromatic number of the complete graphs K_6 , K_{10} , and , in general , K_n .

▷ It would take six colors to color a K_6 graph since every vertex is adjacent to every other

vertex, and we would therefore need a different color for every one. Similarly, it would take ten colors to color the graph K_{10} and n colors to color the graph K_n . Accordingly, $\chi(K_6) = 6$, $\chi(K_{10}) = 10$, and, in general, $\chi(K_n) = n$.

Welch-Powell algorithm for painting a graph G :

1. Order the vertices of G in decreasing degree. (Such an ordering may not be unique since some vertices may have the same degree.)
2. Use one color to paint the first vertex and to paint, in sequential order, each vertex on the list that is not adjacent to a vertex previously painted with this color.
3. Start again at the top of the list and repeat the process painting previously unpainted vertices using a second color.
4. Continue repeating with additional colors until all the vertices have been painted.

Examples :

- Consider the graph G in Fig. 11.13. (a) Use the Welch-Powell algorithm to paint G (referring to colors simply as “a”, “b”, etc.). (b) Find the chromatic number of G .

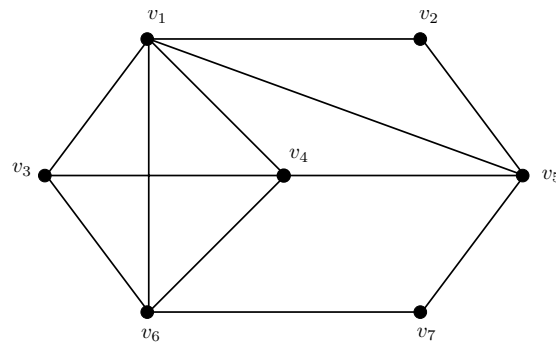


Figure 11.13:

(a) List the vertices of G in order of decreasing degrees as shown below. Assign color a to the first vertex, v_1 . The next vertex on the list not adjacent to v_1 is v_7 : assign color a to v_7 . Now move to color b . Assign color b to the first unpainted vertex, v_4 . The next unpainted vertex not adjacent to v_4 is v_2 : assign color b to v_2 . Since all remaining unpainted vertices are adjacent to v_4 or v_2 , move to color c . Repeat this process with additional colors until all vertices are painted. The completed assignment is listed as follows.

Vertex	v_1	v_2	v_3	v_4	v_5	v_6	v_7
Degree	5	4	4	4	3	3	3
Color	a	b	c	c	b	d	a

(b) The vertices $v_1, v_3, v_4,$ and v_6 are connected to each other and hence must be painted different colors. Thus at least four colors are required to paint G . Since (a) uses only four colors to paint G . $\chi(G) = 4$.

- Repeat previous problem using the graph H in Fig 11.14.

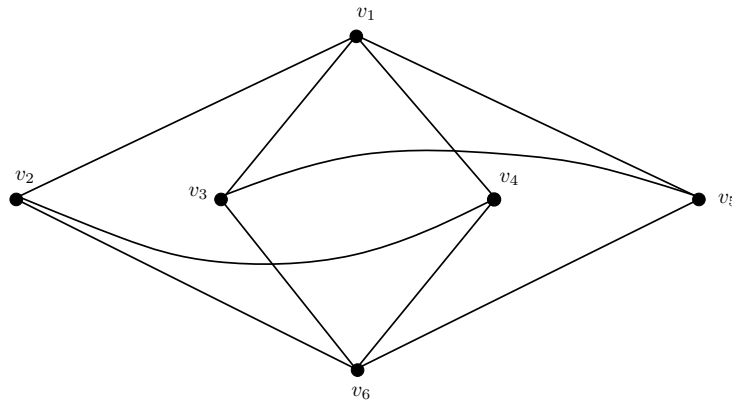


Figure 11.14:

- Repeat previous problem using the graph g in Fig 11.15.

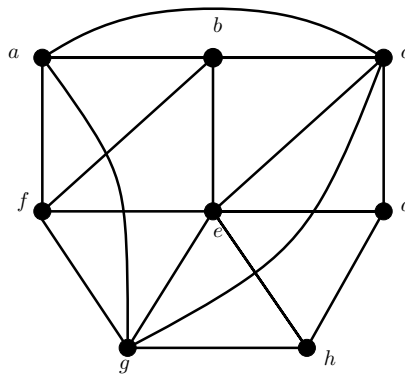


Figure 11.15:

Example : Find the chromatic number of the bipartite graphs $K_{3,4}$ and $K_{2,6}$.

Theorem : Then following are equivalent for a graph G :

- (1) G is 2-colorable.
- (2) G is bipartite.
- (3) Every cycle of G has even length.

Theorem : Any planar graph G is five-colorable.

- **Dual graph G^d** : Given graph G , place a point(vertex) inside each region, including the infinite region, determined by the graph. For each edge shared by two regions, draw an edge connecting the vertices inside these regions.

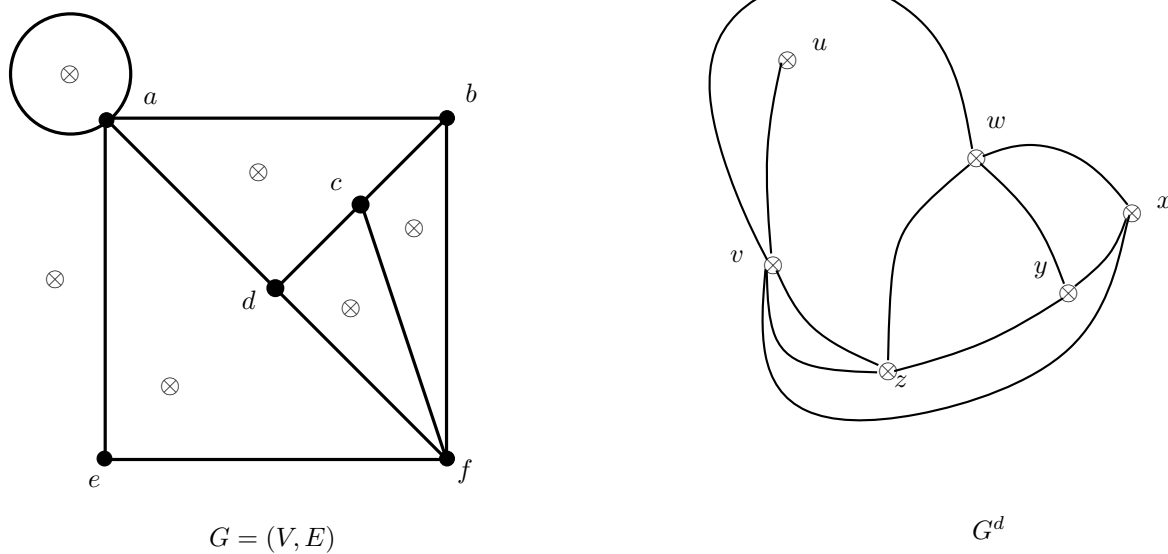


Figure 11.16:

Theorem : Every planar map M is five-colorable.

11.7 Hamilton Paths and Cycles

Definition : If $G = (V, E)$ is a graph or multigraph with $|V| \leq 3$, we say that G has a *Hamilton cycle* if there is a cycle in G that contains every vertex in V . A *Hamilton path* is a path (and not a cycle) in G that contains each vertex.

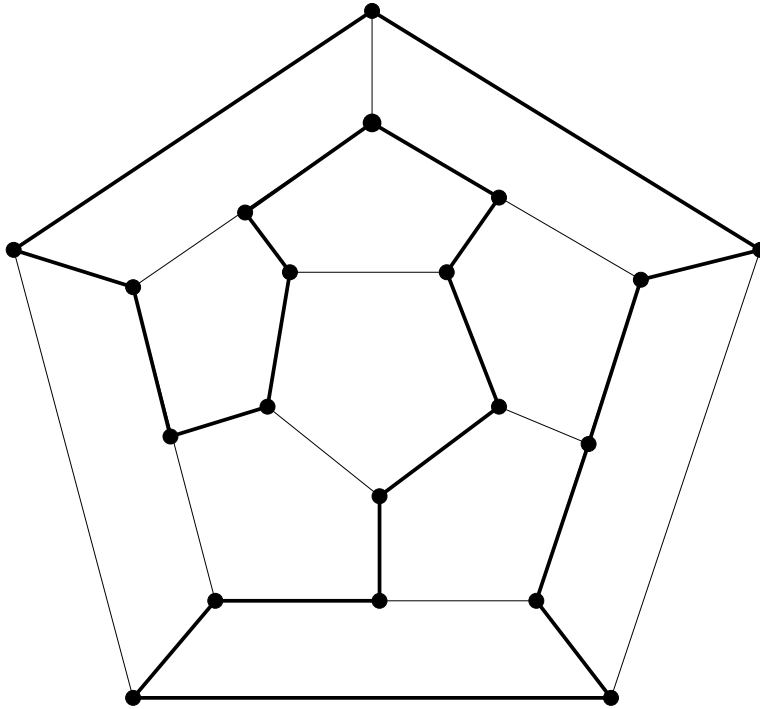


Figure 11.17:

- Helpful hints to find a Hamilton cycle in a graph $G = (V, E)$.
 1. If G has a Hamilton cycle, then for all $v \in V$, $\deg(v) \geq 2$.
 2. If $a \in V$ and $\deg(a) = 2$, then the two edges incident with vertex a must appear in every Hamilton cycle for G .
 3. If $a \in V$ and $\deg(a) > 2$, then as we try to build a Hamilton cycle, once we pass through vertex a , any unused edges incident with a are deleted from further consideration.
 4. In building a Hamilton cycle for G , we cannot obtain a cycle for a subgraph of G unless it contains all the vertices of G .

Theorem : Let $G = (V, E)$ be a loop-free graph with $|V| = n \geq 2$. If $\deg(x) + \deg(y) \geq n - 1$ for all $x, y \in V, x \neq y$, then G has a Hamilton path.

Corollary : Let $G = (V, E)$ be a loop-free graph with $n(\geq 2)$ vertices. If $\deg(v) \geq (n - 1)/2$ for all $v \in V$, then G has a Hamilton path.

Theorem : Let $G = (V, E)$ be a loop-free undirected graph with $|V| = n \geq 3$. If $\deg(x) + \deg(y) \geq n$ for all nonadjacent $x, y \in V$, then G contains a Hamilton cycle.

Corollary : If $G = (V, E)$ is a loop-free undirected graph with $|V| = n \geq 3$, and if $\deg(v) \geq n/2$ for all $v \in V$, then G has a Hamilton cycle.